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ON A TRANSFORMATION OF LAGUERRE.

BY PERCEY F. SMITH.

INTRODUCTION.

IT is proposed in this paper to give an elementary and purely geometrical exposition of a transformation of space, which was for the first time defined and very briefly discussed by Laguerre in the *Comptes Rendus* for 1881, p. 71 :—Sur la transformation par directions réciproques. Although treated more at length by Darboux in the *Leçons sur la théorie générale des surfaces*, Vol. I, p. 253, the development is not systematic, and the properties of the transformation as such are not brought out clearly. In order to do this it is desirable not to adopt the definitions of Laguerre or Darboux, but rather that proposed by E. Müller in the *Monatshefte für Mathematik und Physik*, Vol. IX, 1898 : Ueber die Geometrie orientirter Kugeln nach Grassmann. In this memoir, the methods of the *Ausdehnungslehre* are followed.

For this purpose it becomes necessary to introduce a configuration of spheres, *the spherical complex*. This complex is a special case of the general *Kugelcomplex* of Sophus Lie,* defined as the assemblage of ∞^3 spheres cutting a fixed sphere under constant angle. In the present instance, the fixed sphere becomes a fixed plane, the *fundamental plane* of the spherical complex. Various theorems on this configuration are developed in sections 1 and 2.

The transformation in question is, under the definition of Müller, most fittingly designated : *inversion in a spherical complex*. Planes are transformed into planes, and the ∞^2 planes enveloping a surface F into the ∞^2 planes enveloping a second surface F', so that we say : F inverts into F'. In particular, spheres invert into spheres, and a point regarded as the envelope of all planes passing through it, into a sphere. Thus the transformation is not a point transformation. In order, however, that a given plane may invert into a *unique* plane, it is found necessary to assign a definite *orientation* to that plane, determined, *e. g.* by the direction of a normal. Such a plane is said to be *oriented*.

The configuration of space consisting of a point P and a plane E through P is known as a *surface element*, and may be denoted by (P, E). If a surface

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* First introduced in the well-known memoir : Ueber Complexe, insbesondere Kugelcomplexe, etc., *Math. Annalen*, Vol. 5.

F touches E at P, then (P, E) is said to belong to F. Thus a surface F defines ∞^2 surface elements, and is uniquely defined by them. Two surfaces are tangent if they possess a surface element in common. Surfaces having contact in the element (P, E) go over by any point transformation which transforms P into P' into surfaces having contact in an element (P', E'). For this reason, point transformations are called *contact transformations.* The transformation of Laguerre however belongs to a general class in which it is necessary to consider surface elements as *oriented*, *i. e.* as consisting of a point and an oriented plane through it. A surface F now defines two distinct continuous systems each of ∞^2 oriented surface elements, and if F is regarded as defined by only one of these, F is said to be *oriented.*[*] Two oriented surfaces F and F' are tangent if they possess a common oriented surface element. By inversion in a spherical complex, which inverts an oriented plane E into E', oriented surfaces touching in (P, E) invert into oriented surfaces tangent in (P', E'). As before, the transformation is, then, a *contact transformation.*

The transformation of Laguerre, from its elementary character, deserves to be numbered among the transformations of elementary geometry, viz., collineations, correlations, and inversion in a sphere. In fact, many of its properties developed in the following pages are identical with those of the last mentioned transformation. Inversion in a sphere may indeed be defined with reference to a special *Kugelcomplex*, viz., the ∞^3 spheres cutting a fixed sphere, the sphere of inversion, orthogonally.[†] Then any point P is transformed into a point P' determined by the property that all spheres of the complex passing through P also pass through P'. The two transformations in question are thus clearly associated with the two most immediate special cases of the general *Kugelcomplex.*

It may be remarked that a transformation founded in the same manner on the general *Kugelcomplex* may be established, which is, however, compounded in a simple manner of the two foregoing. All belong to the general class studied by Lie, *l. c.*, of contact transformations by which spheres remain spheres.

Finally, the results of the following sections may be interpreted for the plane, *mutatis mutandis.*[‡]

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[*] Laguerre, *l. c.* uses the term *semi-surface*, and for oriented plane, *semi-plan.*

[†] *Cf.* the memoir by E. Müller, *Monatshefte für Math. u. Phys.* Vol. III.

[‡] Laguerre studied originally the transformation in the plane; *v. Bulletin de la Société Math. de France*, Vol. 8, p. 196.

1. The Spherical Complex. Given in space a plane π and a sphere S of centre c and radius r, then every point c' of space is the centre of a unique sphere S' determined by the condition that the intersection with π of the line joining c and c' shall be a centre of similitude of S and S'.* If c and c' are on the same side of π, this point is an external centre of similitude,—if on opposite sides, an internal centre. This assemblage of ∞^3 spheres may be called a *spherical complex*, and π the *fundamental plane* of the complex. Denoting by C the spherical complex, we may write, C is (S, π).

If S' and S'' are two spheres of C, then, by well-known theorems on centres of similitude, a centre of similitude of S' and S'' will lie in π. Thus *any* sphere of C will serve with π for definition.

The spherical complex C may be constructed also as follows:—let p be any point of π and consider the cone of revolution V with vertex at p and circumscribing S. Then all spheres inscribed in V belong to C, and if V be now displaced by a translation such that p moves in π, the ∞^3 spheres inscribed in V in all of its different positions constitute the spherical complex. In particular, if p describes a straight line l in π, we shall get ∞^2 spheres of C, viz., all spheres touching the two planes passed through l tangent to S, whose centres lie in the plane containing l and the centre of S.

The points of the fundamental plane are to be regarded as spheres of the complex of radius zero.

The plane at infinity may be chosen as fundamental plane, in which case the spherical complex consists of all spheres of the same radius as S.

We may consider the ∞^3 points of space as a spherical complex Ω. Any plane of space will serve as fundamental plane for this complex.

Denoting now by p the perpendicular distance from c, the centre of S, to π, taken with positive or negative sign according as c is on one or the other side of π, then the ratio $p : r$, where r is the radius of S, is numerically the same for all spheres of C. In order to make this ratio retain the same sign, we may assign opposite signs to the radii of the spheres of C according as these spheres lie upon one side or the other of π. Then setting $p : r$ equal to κ, we shall call κ the *constant* of the complex (S, π). Evidently κ and π (*i. e.* four parameters) define the complex.

All spheres of a complex whose constant κ is numerically less than unity cut the fundamental plane under a constant angle $\cos^{-1}\kappa$, which property may also serve as definition of the complex.

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* The treatise of Rouché et Comberousse: *Traité de géométrie élémentaire*, troisième édit., 2° partie, p. 257, may be consulted for the properties of centres of similitude, etc.

In the specialcase $\kappa = \pm 1$, the spheres of the complex touch the fundamental plane. Such a complex will be called *special.* If $\kappa = 0$, the complex consists of all spheres whose centres lie in the fundamental plane, and will be termed an *orthogonal* complex.

The necessity which here appears of attaching a sign to the radius of a sphere is identical at bottom with that of assuming a definite *orientation* for the sphere.

We adopt the convention, *the normal is directed from or toward the centre according as r is positive or negative.*

Two given oriented spheres S and S' with centres c and c' have a *unique* centre of similitude x defined by

$$\frac{cx}{c'x} = \frac{r}{r'},$$

r and r' being the radii of S and S' respectively. Thus x is the external or internal centre of similitude according as S and S' have like or unlike orientations. Then three oriented spheres have a unique axis, and four a unique plane of similitude. Hence *four oriented spheres determine a spherical complex whose fundamental plane is the plane of similitude.*

2. Spheres common to two or more complexes. The preliminary notions just developed render easy the proofs of the following general theorems:

I. *Two complexes have ∞^2 oriented spheres in common, viz. all spheres inscribed in a certain diedral angle whose edge is the intersection of the fundamental planes.*

II. *Three complexes have ∞^1 oriented spheres in common, viz. all spheres inscribed in a certain cone of revolution whose vertex is the point of intersection of the fundamental planes.*

III. *Four complexes have a unique oriented sphere in common.*

Proofs: Let $C_i\ (S_i, \pi_i)$, $i = 1, 2, 3, 4$, be four complexes, and denote by c_i the centre of S_i.

I. We may determine *one* sphere common to C_1 and C_2 thus: Through x, the centre of similitude of S_1 and S_2, draw a line cutting π_1 and π_2 in y_1 and y_2, then the intersection of $c_1 y_1$ and $c_2 y_2$ is the centre of an oriented sphere S common to C_1 and C_2, y_1 being the centre of similitude of S_1 and

S, and y_2 of S_2 and S. Now, *every* sphere inscribed in the diedral angle formed by passing planes through the intersection of π_1 and π_2 tangent to S will belong to C_1 and C_2, and *all* common spheres are thus determined, for the centre of similitude of S and any other common sphere must lie in π_1 and π_2.

II. As before, first determine *one* common sphere. Let S be common to C_1 and C_2, and x the centre of similitude of S and S_3. Draw any line through x and the line of intersection of π_1 and π_2 cutting π_3 in y, then it is easily seen that the line joining y and c_3 intersects the plane of centres of the spheres common to C_1 and C_2 in the centre of an oriented sphere common to C_1, C_2, and C_3. Hence *all* the common spheres are inscribed in the cone of revolution whose vertex is the intersection of π_1, π_2, π_3 and which circumscribes S.

III. The common sphere is thus constructed: Let ν be the cone of revolution circumscribing all spheres common to C_1, C_2, C_3, o its vertex, l its axis. If, now, S is any common sphere of C_1, C_2, and C_3, x the centre of similitude of S and S_4, and y the intersection with π_4 of the line joining x and o, then $y c_4$ intersects l in the centre of an oriented sphere common to C_1, C_2, C_3, and C_4. This common sphere must be unique, for if there were two such, their centre of similitude would be a point common to the four fundamental planes.

Exceptions to the general theorems may arise due to special relations of the given complexes. For example, if π_1 and π_2 coincide, the common spheres of C_1 and C_2 are the points of the common fundamental plane.

3. Inversion in a spherical complex. This is the transformation of Laguerre, a transformation of planes into planes, but not a point transformation. The designation,* inversion in a spherical complex, is peculiarly appropriate, as will appear from the definition of the transformation. Remembering the statements in the *Introduction* regarding oriented surfaces and the contact of such surfaces, the definition of the transformation may be thus stated:

All oriented spheres of a given spherical complex C which touch a given oriented plane are tangent to a second oriented plane. These planes intersect on the fundamental plane of C and are said to be inverse with respect to C.

The truth of the statement follows from the following consideration:

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* *Cf.* Müller, *loc. cit.*

Any two oriented spheres of C touching an oriented plane E have their centre of similitude in the intersection of E and π, hence the second oriented plane through this line tangent to one of the two spheres is tangent also to the other.

The transformation is *involutory*, *i. e.* if E is transformed into E', then conversely, E' transforms into E. Parallel planes remain parallel and the plane at infinity is invariant.

If $\kappa < 1$, the planes cutting π under the angle $180° - \cos^{-1} \kappa$, *i. e.* the planes touching the spheres of C along their intersections with the fundamental plane are unchanged by the transformation.

Special Cases. Inversion in a special complex ($\kappa = 1$) transforms an arbitrary plane into the fundamental plane. This case is of no interest.

Inversion in an orthogonal complex ($\kappa = 0$) is plainly a simple reflection on the fundamental plane.

If the fundamental plane is at infinity, the transformation displaces every plane parallel to itself a distance equal to the constant diameter of the spheres of the complex, and reverses at the same time the orientation. In particular, inversion in Ω, the complex of all points, merely changes the orientation.

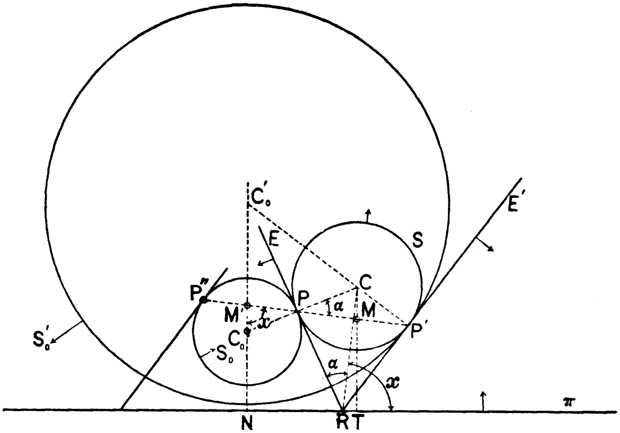

FIG. 1.

4. Transformation of a surface by inversion in a spherical complex. The tangent planes of a surface F are transformed into planes enveloping a surface F'. The surfaces F and F' are said to be inverse with respect to the complex. If no definite orientation is assigned to F, then each of its tangent planes inverts into two distinct tangent planes to F', and F' consists of two nappes.

In particular the following theorem is fundamental:

An oriented sphere S_0 is transformed by inversion in a spherical complex C into an oriented sphere S'_0. The radical plane of S_0 and S'_0 is the fundamental plane of C.

Proof: Referring to fig. 1, let S_0 be the given oriented sphere, with orientation as marked, π the fundamental plane, E any oriented plane tangent to S_0, and E' the inverse of E, S being a sphere of C touching E. Taking the angles x and a and points as marked, we have

$$\frac{\sin\ x}{\sin\ a}=\frac{TC}{CP}=\kappa,$$

where κ is the constant of the complex C. From the equality of the angles in the figure:

$$\kappa=\frac{\sin\ x}{\sin\ a}=\frac{C_0P}{C_0M'}=\frac{C'_0P'}{M'C'_0}=\frac{C_0P+C'_0P'}{C_0M'+M'C'_0}. \tag{1}$$

Since
$$\overline{C'_0R}^2=\overline{C'_0N}^2+\overline{NR}^2=\overline{C'_0N}^2+\overline{C_0R}^2-\overline{C_0N}^2;$$
$$=\overline{C'_0P'}^2+\overline{RP'}^2=\overline{C'_0P'}^2+\overline{C_0R}^2-\overline{C_0P}^2,$$

we get

$$\overline{C'_0N}^2-\overline{C_0N}^2=\overline{C'_0P'}^2-\overline{C_0P}^2. \tag{2}$$

Putting
$$C_0P=-R,\ \ C'_0P'=R',\ \ NC_0=p,\ \ NC'_0=p',$$

(1) and (2) become

$$\kappa=\frac{R'-R}{p'-p},\qquad p'^2-p^2=R'^2-R^2,$$

whence

$$p'=\frac{\kappa^2+1}{\kappa^2-1}\,p-\frac{2\kappa}{\kappa^2-1}\,R,\qquad R'=\frac{2\kappa}{\kappa^2-1}\,p-\frac{\kappa^2+1}{\kappa^2-1}\,R. \tag{A}$$

Since then p' and R' are independent of S, the theorem is proven.

The equations (A) furnish an analytical definition of the transformation. We may remark that a sphere belonging to C (in which case $p:R=\kappa$) inverts into itself.

A point, considered as the locus of all oriented planes passing through it, *i. e.* the limiting case of an oriented sphere with vanishing radius, transforms in general into an oriented sphere. Thus inversion in a spherical complex is not a point-transformation. The points of the fundamental plane are invariant.

When the fundamental plane is at infinity, the transformation is especially simple. Let R be the constant radius of the oriented spheres of such a complex; then an oriented sphere S of radius r inverts into a sphere with the same centre and a radius $= 2R - r$, *i. e.* S is merely *dilated.* In particular, a point is dilated into a sphere of radius $2R$. Any surface is transformed into a parallel surface. The following theorem is also evident:

A non-oriented surface is its own inverse with respect to the complex Ω *of all points.*

The formulas (A) for $\kappa^2 = -1$ define a transformation studied by Bonnet in the *Comptes Rendus*, Vol. 42, p. 485; a point inverts into a sphere whose centre lies in the fundamental plane.

5. Other properties of the transformation.

If (fig. 1) E is tangent to a surface F at P, then E' is tangent to the inverse surface F' at P', while the sphere S of C touches F at P and F' at P'. Hence the theorem:*

All spheres of a given spherical complex C *which touch a surface* F *are tangent also to a surface* F', *the inverse of* F *with respect to* C, *and the tangent planes to* F *and* F' *at these points of contact are inverse with respect to* C.

Two surfaces tangent to E at P transform into surfaces tangent to E' at P'; thus the transformation is a *contact* transformation. We remark here as before, that "tangent" implies coincidence in *orientation* of the tangent planes.

The ∞^1 oriented planes tangent to a surface F along any curve c invert into planes tangent to the inverse surface F' along a curve c'. In this sense, curves on F are transformed into curves on F'. In particular, *the lines of curvature of* F *go over into lines of curvature on* F'. For let P and P_1 be consecutive points on a line of curvature of F through P, then one of the *principal spheres* of F touches F at P and *also* at P_1; hence this sphere inverts into a principal sphere of F' touching at P' and P_1', consecutive points of a line of curvature on F'.

**Cf.* Darboux, *l. c.*

If the oriented plane E touches two surfaces F and G in P and Q, then PQ may be called the *tangential distance* of F and G in E. Now let E transform into E' tangent to the inverse surfaces F' and G' in P' and Q'; then $PQ = P'Q'$. For a sphere S of C touches F and F' in P and P', and a second sphere S' of C is tangent to G and G' in Q and Q'; then the lines PQ and $P'Q'$ are common tangents to S and S' through their centre of similitude. We have then the theorem:

*Tangential distance is unaltered by the transformation.**

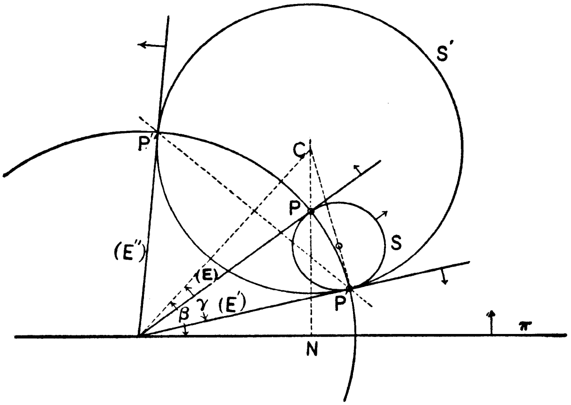

FIG. 2.

Turning to formulas (A) and fig. 2, let $NP = p$, then the point P inverts into a sphere S' of radius R' such that

$$p' = NC = \frac{\kappa^2+1}{\kappa^2-1} NP, \quad R' = \frac{2\kappa}{\kappa^2-1} p = \frac{2\kappa}{\kappa^2+1} p'.$$

If then P describes an oriented surface F, S' will touch the inverse surface F', and the centre of S' will describe a surface F_0 obtained from F by increasing its ordinates relative to π in the constant ratio $\kappa^2 - 1 : \kappa^2 + 1$, or if we set, for convenience, $\mu = 2\kappa/(\kappa^2 - 1)$, $\sqrt{1-\mu^2} : 1$.

We remark also, since $p' : R' = 1 : \mu$, that the spheres S' belong to a spherical complex with π for fundamental plane.

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*The paper of Müller, *l. c.*, may be consulted for analogues of the well-known theorems of Bonnet and Joachimsthal on lines of curvature, in which tangential distance replaces angle of intersection.

If, now, F is not oriented, S' will touch F' in P' and P'', points symmetrical with respect to the plane E touching F_0 at c, and S' will *envelop* F', hence the theorems:

The inverse of an oriented surface F is touched by all spheres belonging to a spherical complex with fundamental plane π and constant $1 : \mu$ *whose centres lie on a surface F_0 derived from F by increasing its ordinates relative to the fundamental plane in a constant ratio*

$$\sqrt{1-\mu^2} \; : \; 1\,;$$

if F is not oriented, then the inverse is the envelope of these spheres, the two points of contact being symmetrical with respect to the plane touching F_0 in the centre of the sphere.

This last theorem is given in somewhat altered form by Laguerre and Darboux. The surfaces F and F_0 intersect π in the same curve. In particular, if F is an algebraic surface, so is F_0, and of the same order as F.

6. Other definitions of the transformation. Let S be an oriented sphere and E any oriented plane, and let that tangent plane to S which is parallel to E and has the same orientation touch S at P; then P is called the *spherical image* of E on S. Evidently P is the spherical image of ∞^1 oriented planes.

Referring to fig. 1, the line joining the spherical images on S of E and E' passes through M, the pole of π with respect to S. Hence P and P' are corresponding points in a spherical inversion on the centre M which inverts S into itself. Let, now, F be *any* oriented plane intersecting π in l, and Q its image on S; let QM intersect S again in Q', then the inverse of F is the oriented plane through l whose spherical image on S is Q'.

Hence we have the result:

Inversion on a spherical complex may be thus defined:— Corresponding planes intersect on the fundamental plane, and their spherical images on any sphere of the complex are inverse points in a spherical inversion whose centre is the pole of the fundamental plane with respect to that sphere.

Again in fig. 1, the line PP' intersects c_0c_0' in M', the centre of similitude of S_0 and S_0'. For, with the previous notation,

$$\frac{C_0M'}{C_0'M'} = \frac{R}{R'}.$$

If, now, PP' be produced to meet S_0 again at P'', then P'' is the spherical image of E' on S_0. Now P and P' are corresponding points in a spherical inversion on the centre M'. Hence, as before, a new definition :

*Corresponding planes intersect on the fundamental plane and their spherical images on any oriented sphere are inverse points in a spherical inversion whose centre is the centre of similitude of the given sphere and its inverse with respect to the complex.**

The spherical image on a sphere of any surface F is the locus of the images of the planes tangent to F. Hence this theorem :

The spherical images on any sphere S of two surfaces inverse with respect to a spherical complex are transformed into each other by a spherical inversion.

7. Transformation of a spherical complex. Having seen in what way the familiar configurations of space are transformed, we may now consider the transformation of a spherical complex.

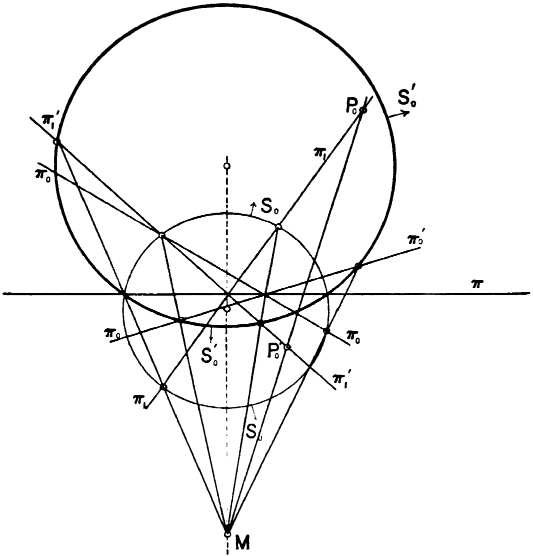

FIG. 3.

Let the spherical complex $C_0(S_0, \pi_0')$ be inverted on the complex C with fundamental plane π. Then S_0 inverts into S_0', and M is the centre of similitude of S_0 and S_0'. Now S_0 may be transformed into S_0' by spherical inversion on the centre M. By well-known theorems, there is a unique homology (T) with centre M and plane π, which transforms S_0 into S_0' exactly as does the above spherical inversion. Hence by the previous section, (T) transforms a plane tangent to S_0 into a plane tangent to S_0' *exactly as the inversion in* C. Let π_0 transform into π_0' by (T). Then any cone of revolution circumscribing S_0, with its vertex in π_0, transforms by (T) or by the inversion in C into a cone of revolution circumscribing S_0' with vertex in π_0'. Hence the theorem:

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* Given by Darboux, *l.c.*, but the relation of the centre of inversion to the given sphere is not indicated.

A spherical complex C_0 *is transformed by inversion in a spherical complex* C *into a spherical complex* C_0'. *The fundamental planes of* C_0 *and* C_0' *intersect on the fundamental plane of* C. *If the spheres* S_0 *of* C_0 *and* S_0' *of* C_0' *correspond, then the circles of intersection of* π_0 *with* S_0 *and of* π_0' *with* S_0' *are inverse in a spherical inversion whose centre is the centre of similitude of* S_0 *and* S_0'.

A special complex $(\kappa = 1)$ inverts into a special complex. In particular, the complex Ω of all points transforms into a complex C' with the same fundamental plane as C, and the constant of C' is $(\kappa^2 + 1)/2\kappa$ (*cf.* §5).

The following theorem given by Darboux (p. 255) is now established: *If a surface* F *is enveloped by spheres belonging to a spherical complex* C_0, *then these spheres invert into spheres belonging to a complex* C' *and envelop a surface* F', *the inverse of* F.

8. Simultaneous Invariant. Involution. Let the complexes C_0 (S_0, π_0), C_1 (S_0, π_1) (fig. 3) invert on C into C_0' (S_0', π_0') and C_1' (S_0', π_1'). Let P_0 and P_0' be the poles of π_0 and π_0' with respect to S_0 and S_0' respectively. Then since π_0 and π_1 are transformed into π_0' and π_1' by the collineation (T) which transforms S_0 into S_0', if P_0 lies in π_1 then will P_0' lie in π_1', *i. e. two complexes whose fundamental planes are conjugate with respect to a common sphere invert into two complexes having the same property.** This is then an *invariant* property of two complexes. A simple calculation gives as the analytic condition for this relation

$$\cos\theta + \kappa_0\kappa_1 = 0,$$

where θ is the angle between the fundamental planes, and κ_0 and κ_1 the constants of the two complexes. The expression $\cos\theta + \kappa_0\kappa_1$ may be called the simultaneous invariant of the two complexes.

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*If π_0 and π_1 are conjugate with respect to *one* of the ∞^2 spheres common to C_0 and C_1, they are conjugate with respect to *every* such sphere.

Two complexes are said to be *conjugate* or in *involution* when their simultaneous invariant vanishes.

If C_0 is an orthogonal complex, then every complex whose fundamental plane is perpendicular to π_0 is in involution with C_0. If π_0 is the plane at infinity then the system of ∞^3 complexes in involution with C_0 is the system of all orthogonal complexes. This is true in particular for the complex of all points Ω, *i. e. any orthogonal complex and Ω are in involution.*

9. System of four spherical complexes mutually in involution. If S is the oriented sphere common to C_1, C_2, C_3, C_4, and if the fundamental planes π_1, π_2, π_3, π_4 form a self-conjugate tetrahedron with respect to S, then C_1, C_2, C_3, C_4 are two by two in involution.

In particular three orthogonal complexes whose planes are mutually perpendicular and the complex Ω form a system of four complexes mutually in involution. By inversion in any spherical complex this system is transformed into a general system in involution, the point of intersection of the three planes inverting into the sphere S, common to the four transformed complexes.

10. The inverse of a central quadric. We may now apply the preceding results to the discussion of the oriented surface F obtained by inversion on a spherical complex of a non-oriented central quadric f.

Since f is sibi-reciprocal (*i. e.* its own inverse) with respect to four spherical complexes mutually in involution, viz. orthogonal complexes on the three central planes and the complex of all points Ω, then:

Theorem I. The surface F is sibi-reciprocal with respect to four spherical complexes mutually in involution.

Let these four complexes be C_1, C_2, C_3, C_4, with four fundamental planes π_1, π_2, π_3, π_4, and suppose *e. g.* C_1 is the inverse of Ω. Then by the theorem at the end of §5 we see that F is the envelope of ∞^2 spheres whose centres lie on a quadric F_1 and which belong to C_1. But no one of the four complexes C_i is to be distinguished, since, by the preceding section, they form a general system in involution, hence we have:

Theorem II. The surface F is the envelope in four ways of ∞^2 spheres whose centres lie on a quadric F_i and which belong to a spherical complex C_i.

Then by the same theorem:

Theorem III. With each quadric F_i is associated a second quadric f_i derived from F_i by reducing its ordinates relative to π_i in a constant ratio, and F and f_i are inverse with respect to C_i.

It is easy to show that the four quadrics F_i are confocal. In fig. 2, let c be a point on F_i and E the tangent plane, then P' and P'' are on F, and E' and E'' are tangent planes to F. Now if E is a minimum plane,* then the line $P'P''$ being perpendicular to E must lie in E, hence E is tangent to F, and in the same way if E' is a minimum plane, E and E'' must coincide with it,† and therefore F_i and F are inscribed in the same minimum developable, hence:

Theorem IV. The four quadrics F_i and the surface F are confocal.

We may next determine the class of F. Consider any pencil of oriented planes; this inverts into the planes of a cone of revolution,‡ four of which are tangent to f_i, hence:

Theorem V. The surface F is of the fourth class.

Since a tangent plane to f_i at a point of its intersection with π_i inverts into two planes touching F at this same point, we have:

Theorem VI. The surface F contains four double conics, viz., the intersection of f_i (or F_i) with the fundamental planes of the C_i.

Let l be a minimum line at infinity; through this line may be passed two minimum planes tangent to F_i and hence to F. Since F is of the fourth class, from the remark of the preceding note, follows:

Theorem VII. The surface F contains the imaginary circle at infinity as double line.

The *singular* focal (or minimum) developable of a surface containing the imaginary circle at infinity is the developable of minimum planes touching the surface along this circle, and the double lines of this developable are the singular focal lines of the surface, hence:

Theorem VIII. The singular focal lines of F are the focal lines of the quadrics F_i, and these are the only focal lines of F.

The sphere belongs to the above class of surfaces, *whose minimum developable and singular minimum developable are identical.*

The preceding results except the last theorem are given by Darboux and Laguerre without proof in the papers already mentioned. We return to the surface F in section 15.

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* *i. e.* E is tangent to the imaginary circle at infinity.

† If E' is a minimum plane, then CP being perpendicular to E' lies in E', *i. e.* E' passes through the centre of S', and therefore P' lies at infinity on the imaginary circle, *i. e. a minimum plane tangent to F touches F on the imaginary circle at infinity.*

‡ The vertex is the point of intersection of the axis of the pencil with π_i.

11. Pencil of spherical complexes. The system of ∞^1 complexes containing all ∞^2 spheres common to two given complexes C_0 and C_1 are said to form the *pencil* (C_0, C_1). Since the point spheres common to C_0 and C_1 are the points of intersection of the fundamental planes π_0 and π_1, the fundamental planes of the pencil (C_0, C_1) belong to the pencil of planes (π_0, π_1). Between the pencils (π_0, π_1), (C_0, C_1) exists a $(1, 1)$ correspondence.

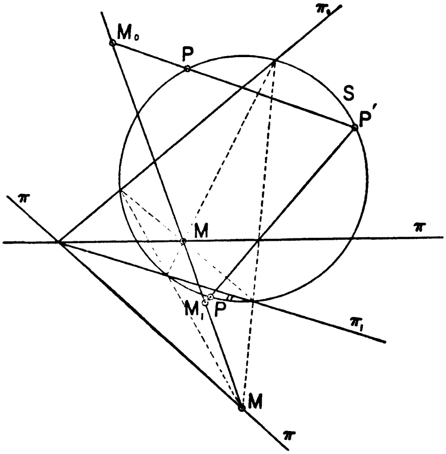

FIG. 4.

Let the sphere S (fig. 4) belong to C_0 and C_1, then there are two special complexes in (C_0, C_1); the fundamental planes of these special complexes touch S, and the common spheres of C_0 and C_1 consist of all oriented spheres touching these planes. One orthogonal complex belongs to (C_0, C_1); its fundamental plane is the locus of centres of these spheres.

If π_0 and π_1 coincide, the pencil becomes all complexes with the fundamental plane π_0.

12. Transformation of two given spherical complexes into each other. In fig. 3 if S_0 becomes a sphere of C, S_0' becomes identical with S_0, and M becomes the pole of π with respect to S_0. The collineation (T) now becomes the *harmonic* projection with centre M and plane π.

If then, in fig. 4, M and M' are the two centres with respect to which the intersections of π_0 and π_1 with S are inverse,* then C_0 and C_1 are inverse with respect to *each* of two complexes of the pencil (C_0, C_1) and the fundamental planes of these complexes pass through M and M'. Denoting these complexes by C and C' and their fundamental planes by π and π', then the pencil $\pi' \pi_1 \pi \pi_0$ is harmonic, and C and C' are in involution. That is

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* M and M' are both real only if $|\kappa_0|$ and $|\kappa_1|$ both exceed or are both less than unity.

Two spherical complexes are inverse with respect to each of two complexes of the pencil determined by them, and these two complexes are in involution.

Some special cases require explanation:

(1) *π_0 and π_1 are coincident.*

Dividing the equations (A), we get

$$\frac{p'}{R'} = \left(\lambda \frac{p}{R} - 1\right) \div \left(\frac{p}{R} - \lambda\right),$$

where

$$\lambda = \frac{\kappa^2 + 1}{2\kappa}.$$

Hence the complexes with constants $p : R$ and $p' : R'$ are inverse with respect to the two complexes with fundamental plane π and whose constants are the solutions of $\kappa^2 - 2\kappa\lambda + 1 = 0$. The product of the roots here being $+1$, the two complexes are in involution with $\theta = 180°$.

(2) *Ω one of the given complexes.*

By the foregoing, Ω inverts on C into a complex C_0 with the same fundamental plane as C and with constant

$$\kappa_0 = \frac{\kappa^2 + 1}{2\kappa}.$$

This equation has two solutions for κ, with product $= +1$, as before.

The following theorem is important:

A spherical complex C conjugate to each of two given complexes is conjugate to every complex of the pencil determined by them.

For if S is a sphere common to the three complexes, the pole of the fundamental plane of C with respect to S lies in the intersection of the fundamental planes of the two given complexes.

13. Transformations resulting from successive inversions.

We establish in this section certain theorems analogous to well-known theorems on spherical inversions. The former theorems may be established directly, but are most simply derived from the latter by means of the definitions of section 6.

The theorems on spherical inversions are as follows:*

Two successive spherical inversions may always be replaced in ∞^1 ways by two other spherical inversions; the four spheres of inversion belong to the same coaxial system, and the centre of one of the latter inversions may be chosen arbitrarily in the axis.

Two spherical inversions are commutative when and only when the spheres of inversion cut orthogonally.

Consider (fig. 4) the complexes C_0 (S, π_0) and $C_1(S, \pi_1)$. A plane touching S at P inverts in C_0 into a plane touching S at P', and the latter plane inverts in C_1 into a plane tangent to S at P'' (§6), where P is transformed into P'' by successive spherical inversions with centres M_0 and M_1. The spheres of inversion cut S orthogonally, and the product of these inversions may be replaced by two other inversions with centres on the line $M_0 M_1$, whose spheres of inversion cut S orthogonally. One of these centres may be chosen arbitrarily. If, then, M_0' and M_1' are two such centres and π_0' and π_1' their polar planes with respect to S, successive inversions in C_0 and C_1 lead to the same result as successive inversions in C_0' (S, π_0') and C_1' (S, π_1'). For this is plainly true for all oriented planes tangent to S, hence also for the cone of revolution circumscribing S and *any* sphere σ, and from the invariance of tangential distance, σ must transform into the same sphere in both instances.

Denoting, now, the inversion in C by (C), and successive inversions in C and C' by $(C\,C')$, we have

$$(C_0 C_1) = (C_0' C_1').$$

It is easy to see that C_0 being chosen, C_1' is unique. Hence the theorem:

The transformation of space (T) *which is compounded of successive inversions in two spherical complexes, may be resolved in* ∞^1 *ways into successive inversions in two other spherical complexes. The four complexes belong to the same pencil; and, of the two latter, one may be chosen arbitrarily and then the other is uniquely determined.*†

(The proof however fails when π_0 and π_1 coincide. In this case, let S be *any* sphere; then using §6, the same considerations will apply.)

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* *Cf. e. g.* Borel in the Note in Niewenglowski's *Cours de Géométrie analytique.* Vol. III, p. 505.

† An interesting case occurs when C'_0 is chosen as C_1, *i. e.* $(C_0C_1) = (C_1\,C)$. Then C_1 is one of the two complexes of §12, *i. e.* C_0 and C are inverse with respect to C_1, which relation we may denote by $C_0(C)\,C_1$, or $C_1\,(C)\,C_0$. For, if $C_0(C_0C_1)\,C' = C_0\,(C_0)\,C_0\,(C_1)\,C'$, then since $C_0\,(C_1)\,C'\,(C)\,C'$, C' must be C itself.

Choosing the orthogonal complex of the pencil, we get the very important theorem:

The transformation (T) *may be uniquely resolved into a reflection and inversion, or an inversion followed by this reflection.*[*]

If C_0 and C_1 are in involution, and S is any common sphere, the spheres of inversion with the centres M_0 and M_1 will cut orthogonally, and since the spherical inversions with centres M_0 and M_1' are commutative, we have

Inversions in spherical complexes in involution are commutative, and conversely.

An important case is the special case above remarked, when the fundamental planes coincide. Consider, then, two complexes in involution, C and C', with coincident fundamental planes. Since the simultaneous invariant vanishes, the constant of one must be numerically equal to the reciprocal of the constant of the other. Denote the inversions by (C) and (C'); then resolving (CC') into an inversion and reflection, since the common spheres of C and C' are in this case the points of the fundamental plane, the complex in involution with the orthogonal complex is the complex of all points Ω. Calling the reflection (I), we have then

$$(CC') = (\Omega I),$$

or

$$(CCC') = (C\Omega I),$$

and since (CC) is the identical transformation,

$$(C') = (C\Omega I) = (I\Omega C);$$

whence the important theorem:

Inversion in a spherical complex with constant κ *may be resolved into a reflection in the fundamental plane of that complex, change of orientation and inversion in a spherical complex with constant* $1/\kappa$, *or into the same three inversions in the reverse order.*

14. Group of the geometry of reciprocal directions. Representing as before the inversion in the spherical complex C_i by (C_i), from the theorems of the preceding section, the transformation made up of successive inversions in the n complexes, $C_1, C_2, C_3, \ldots C_n$, may be resolved thus:

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[*] *Cf.* Borel, *l. c*, for the corresponding result when (T) is compounded of two spherical inversions. The transformation (T) belongs to the class studied by Wiener in an important paper: Ueber die aus zwei Spiegelungen zusammengesetzten Verwandtschaften, *Leipziger Berichte* 1891, p. 644, in which many general theorems are established.

$$(C_1 C_2 C_3 \dots C_n) = (I_1 I_2 I_3 \dots C) = (C' I'_1 I'_2 \dots I'_n),$$

where the (I)s are reflections. If n is *even*, then from the last theorem of §13,

$$(C) = (I \, \Omega \, C_1), \quad (C') = (C'_1 \, \Omega \, I'),$$

and since an *even* number of reflections is a displacement of space, and (Ω) is change in orientation, we have the result:

A transformation (T) of space which is compounded of successive inversions in any number of spherical complexes belongs to one of two classes:

$$\begin{aligned} &(a) \qquad (T) = (CD) = (D'C') \\ &(b) \qquad (T) = (C \, \Omega \, D) = (D' \, \Omega \, C'), \end{aligned}$$

where (D), (D') are displacements, (C), (C') inversions in spherical complexes, and (Ω) change in orientation.[*]

Consider, now, the system of transformations embracing

(I) all displacements (D), and the identical transformation,
(II) all inversions in spherical complexes (C),
(III) all transformations (CD),
(IV) all transformations $(C \, \Omega \, D)$.

These transformations form a group. To prove this, we may, by virtue of the preceding theorems, confine ourselves to compounding transformations in classes (III) and (IV). Consider, *e. g.*, the result of compounding two transformations of (IV). By applying the preceding theorems, we easily get

$$(C \, \Omega \, D)(C' \, \Omega \, D') = (C \, D \, \Omega \, \Omega \, D' \, C'') = (C D'' C'') = (C \, C''' D'') = (C_1 \, \Omega \, D''').$$

Similarly for two transformations one each of (III) and (IV), or two of (III).

The most general transformation of this group depends upon ten parameters, viz., the inversion on four, the displacement on six. As is well known, the group of the Geometry of Reciprocal Radii, in which spherical inversion plays the same rôle as inversion in a spherical complex in the above group, is a ten parameter group, the most general transformation being composed of a spherical inversion and a displacement. This group embraces *all conformal point transformations* of space. We may then properly call this new group, using Laguerre's designation, the group of the Geometry of Reciprocal Directions. *This group embraces all plane-transformations preserving tangential distances.*[*]

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[*] This theorem implies that a transformation (CD) may be resolved into the second form $(D'C')$. For if we resolve (D) into four plane reflections $(I_1 I_2 I_3 I_4)$, then $(C D) = (C I_1 I_2 I_3 I_4)$, and applying the second theorem of §13, $(C I_1 I_2 I_3 I_4) = (I_1 C_1 I_2 I_3 I_4)$, or finally $= (DC')$. Similarly for (b).

It is interesting and important to note that the group of the Geometry of Reciprocal Directions is simply isomorphic with the group of all displacements and symmetry transformations in space of four dimensions. This is brought about by a (1, 1) correspondence between points in such a space and oriented spheres, as indicated, *e. g.*, in Klein's *Höhere Geometrie*, pp. 472, seq.

15. The correlative of the Cyclide. As is well known, the general quadratic point-manifoldness in the Geometry of Reciprocal Radii is the cyclide, a surface of order four containing the circle at infinity as double line. In the Geometry of Reciprocal Directions, the surface F of class four and containing the circle at infinity as double line, discussed in Section 10, is the quadratic plane-manifoldness. For the cyclide, theorems analogous to theorems I, II and IV hold.[†] Furthermore, the centre of each of the five mutually orthogonal spheres, with respect to each of which the cyclide is its own inverse, is the vertex of a quadric cone enveloped by planes bitangent to the cyclide. On F lie five double conics, viz. in the fundamental planes of the C_i and in the plane at infinity. In fact, *the cyclide and the surface F have correlative properties. Thus F is of order twelve, and is the general surface of class four having the circle at infinity for double line.*

We may then establish the following relation between the central quadric and general cyclide:

A central quadric may be transformed into a cyclide by an inversion in a spherical complex followed by a dualistic transformation.

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[*] This group was first noted by Sophus Lie, p. 186, note. *Cf.* also the Note by Stéphanos in the *Comptes Rendus* for 1881, p. 1195.

[†] *Cf. e. g.* Darboux, *Sur une classe remarquable de courbes et de surfaces algébriques*, p. 116, seq., p. 153.